24 Solving nonhomogeneous ODE systems with constant coefficients

Consider the nonhomogeneous system with constant coefficients

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{f}(t), \quad \boldsymbol{A} = [a_{ij}]_{n\times n}, \quad \boldsymbol{f}\colon \mathbf{R} \to \mathbf{R}^n. \tag{1}$$

Similarly to the case of linear ODE of the n-th order, it is true that

Proposition 1. *The general solution to system* (1) *is given by the sum of the general solution to the homogeneous system plus a particular solution to the nonhomogeneous one:*

$$\boldsymbol{y}(t) = \boldsymbol{y}_h(t) + \boldsymbol{y}_p(t).$$

The proof is left an an exercise and relies on the fact that if $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ solve (1) then $\boldsymbol{y}_1 - \boldsymbol{y}_2$ solves homogeneous system.

Therefore, to solve system (1) we need somehow find a particular solution to the nonhomogeneous system and use the technique from the previous lectures to obtain solution to the homogeneous system. Recall that for the linear equations we considered three approaches to solve nonhomogeneous equations: the variation of the constant (or variation of the parameter) method, the method of an educated guess, and the Laplace transform method. Similarly, we can use the same methods here. We will start with the method of an educated guess. Then we will recall how the Laplace transform can be used for solve ODE. And finally, the most important theoretically method, the variation of the constants method, will be covered at the end.

24.1 Method of an educated guess

Sometimes it is possible to guess what is the form of a particular solution. Consider, e.g., system of the form

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{w}e^{at},$$

where $\boldsymbol{w}$ is a constant vector, and a is a constant. Let us look for a particular solution in the form

$$\boldsymbol{y}_p(t) = \boldsymbol{x}e^{at}.$$

We find that

$$a\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w} \implies (\boldsymbol{A} - a\boldsymbol{I})\boldsymbol{x} = \boldsymbol{w}.$$

Assuming that a is not an eigenvalue of matrix $\boldsymbol{A}$, we can solve the last system

$$\boldsymbol{x} = (\boldsymbol{A} - a\boldsymbol{I})^{-1}\boldsymbol{w}.$$

Example 2. Solve

$$\dot{\boldsymbol{y}} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}.$$

As usual we find that matrix $\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with the eigenvectors $\boldsymbol{v}_1 = (1,1)^\top$ and $\boldsymbol{v}_2 = (3,1)^\top$, therefore, the general solution to the homogeneous system is

$$\boldsymbol{y}_h(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t},$$

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where C_1, C_2 are arbitrary constants. Since 3 is not an eigenvalue of $\boldsymbol{A}$, we can look for a particular solution to the nonhomogeneous system in the form

$$\boldsymbol{y}_p(t) = \begin{bmatrix} A \\ B \end{bmatrix} e^{3t},$$

where A and B are constant to be determined. After plugging $\boldsymbol{y}_p(t)$ and canceling all the exponents, we obtain the system

$$\begin{aligned} 3A &= 2A - B + 1, \\ 3B &= 3A - 2B + 1, \end{aligned}$$

which has the unique solution $A = 1/2, B = 1/2$. Therefore, the general solution to our system is given by

$$\boldsymbol{y}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}.$$

Remark 3. Sometimes the system of the form

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{w}$$

needs to be solved, where $\boldsymbol{w}$ is a given constant vector. Of course, one can "guess" that the particular solution in this case is also constant, i.e., $\boldsymbol{y} = \boldsymbol{u}$ and find that $\boldsymbol{u}$ must satisfy the linear algebraic equation

$$\boldsymbol{A}\boldsymbol{u} + \boldsymbol{w} = 0,$$

which will have a unique solution if $\boldsymbol{A}$ is invertible (that is, if 0 is not an eigenvalue of this matrix). From a slightly more involved theoretical perspective we can look at $\boldsymbol{u}$ as an equilibrium of the original system (at this point the vector field vanishes), and hence we can talk about stability of this equilibrium and the corresponding phase portrait. While it is almost obvious that exactly the same situations are possible (saddle, or node, or focus, etc), the full justification comes from the change of variables:

$$\boldsymbol{y}(t) = \boldsymbol{x}(t) - \boldsymbol{A}^{-1}\boldsymbol{w},$$

which yields

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x},$$

which we studied in detail.

24.2 Using the Laplace transform

We can also use the Laplace transform to attack nonhomogeneous problems. Assume that we need to solve system (1) and let $\boldsymbol{Y}(s)$ be the Laplace transform of $\boldsymbol{y}(t)$:

$$\boldsymbol{Y}(s) = \mathscr{L}\left\{\boldsymbol{y}(t)\right\}.$$

We also will need the fact that

$$\mathscr{L}\left\{\dot{\boldsymbol{y}}(t)\right\} = s\boldsymbol{Y}(s) - \boldsymbol{y}_0.$$

Now by applying the Laplace transform to the left and right hand sides of (1), we find

$$s\boldsymbol{Y}(s) - \boldsymbol{y}_0 = \boldsymbol{A}\boldsymbol{Y}(s) + \boldsymbol{F}(s),$$

or

$$\boldsymbol{Y}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{y}_0 + \boldsymbol{F}(s)),$$

which gives us the formal solution to the problems, provided that we are able to find the inverse Laplace transform

$$\boldsymbol{y}(t) = \mathscr{L}^{-1}\left\{\boldsymbol{Y}(s)\right\}.$$

Example 4. Consider again

$$\dot{\boldsymbol{y}} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By applying the Laplace transform, we find

$$\begin{bmatrix} s-3 & 4 \\ -1 & s+1 \end{bmatrix} \boldsymbol{Y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s-1} \end{bmatrix},$$

from where the vector $\boldsymbol{Y}(s)$ can be found (using any method you prefer to solve the linear system) as

$$\boldsymbol{Y}(s) = \begin{bmatrix} \frac{s-3}{(s-1)^2} + \frac{s-3}{(s-1)^3} \\ \frac{s-2}{(s-1)^2} + \frac{s-2}{(s-1)^3} \end{bmatrix}.$$

Using partial fraction decomposition, we find, e.g., for the first element that

$$\frac{s^2 - 4s + 3 + s - 3}{(s-1)^3} = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3},$$

therefore,

$$\mathscr{L}^{-1}\left\{\cdot\right\} = e^t - te^t - t^2 e^t,$$

and similarly for the second element

$$\mathscr{L}^{-1}\left\{\cdot\right\} = e^t - \frac{1}{2}t^2 e^t.$$

Therefore, the final answer is, as before,

$$\boldsymbol{y}(t) = \begin{bmatrix} 1 - t - t^2 \\ 1 - \frac{1}{2}t^2 \end{bmatrix} e^t.$$

24.3 Variation of the constants

Let $\boldsymbol{\Phi}(t)$ be a fundamental matrix solution to the homogeneous system

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}. \tag{2}$$

Hence, the general solution can be written as $\boldsymbol{y}_h(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}$, where $\boldsymbol{c} = (C_1, \ldots, C_n)^\top$ is a vector of arbitrary constants. *Assume* that this vector is not constant, but a function of t:

$$\boldsymbol{c} = \boldsymbol{c}(t),$$

and plug $\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}(t)$ into (1). We find

$$\dot{\boldsymbol{\Phi}}(t)\boldsymbol{c}(t) + \boldsymbol{\Phi}(t)\dot{\boldsymbol{c}}(t) = \boldsymbol{A}\boldsymbol{\Phi}(t)\boldsymbol{c}(t) + \boldsymbol{f}(t),$$

and since $\dot{\boldsymbol{\Phi}}(t) = \boldsymbol{A}\boldsymbol{\Phi}(t)$, then, finally,

$$\dot{\boldsymbol{c}}(t) = \boldsymbol{\Phi}^{-1}(t)\boldsymbol{f}(t),$$

which has the solution

$$\boldsymbol{c}(t) = \boldsymbol{c}_0 + \int_{t_0}^{t} \boldsymbol{\Phi}^{-1}(\tau)\boldsymbol{f}(\tau)\,\mathrm{d}\tau,$$

where now $\boldsymbol{c}_0$ is the vector of arbitrary constants. Using this solution, we obtain

$$\boldsymbol{y}(t) = \underbrace{\boldsymbol{\Phi}(t)\boldsymbol{c}_0}_{\boldsymbol{y}_h(t)} + \underbrace{\boldsymbol{\Phi}(t)\int_{t_0}^{t} \boldsymbol{\Phi}^{-1}(\tau)\boldsymbol{f}(\tau)\,\mathrm{d}\tau}_{\boldsymbol{y}_p(t)}.$$

This formula is very convenient theoretically, but in actual calculations usually requires quite a few steps. Since the matrix exponent is a special fundamental matrix, we can rewrite the last solution in the form

$$\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{c}_0 + \int_{t_0}^{t} \boldsymbol{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{f}(\tau)\,\mathrm{d}\tau.$$

This last formula requires only a slight modification if we are given the initial conditions $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$:

$$\boldsymbol{y}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{f}(\tau)\,\mathrm{d}\tau.$$

Example 5. Solve

$$\dot{\boldsymbol{y}} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- *Eigenvalues and eigenvectors.* We have

  $$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{bmatrix} 3-\lambda & -4 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2,$$

  hence we have one eigenvalue $\lambda = 1$ multiplicity two. Consider

  $$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

  which implies that the eigenvalue $\lambda = 1$ has only one eigenvector $\boldsymbol{v}_1 = (2, 1)^\top$, and hence our first particular solution to the homogeneous system can be written as

  $$\boldsymbol{y}_1(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t.$$

To find the second linearly independent solution, we will need to look for the generalized eigenvector that solves

$$(\boldsymbol{A}-\lambda\boldsymbol{I})^2\boldsymbol{v}_2=0.$$

We find that

$$(\boldsymbol{A}-\lambda\boldsymbol{I})^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},$$

therefore any vector can be taken as a generalized eigenvector, however, we need to remember that we need to take one, which is linearly independent of $\boldsymbol{v}_1$. I choose to take

$$\boldsymbol{v}_2=\begin{bmatrix}1\\ 0\end{bmatrix}.$$

Therefore, the second particular solution to the linear system is

$$\boldsymbol{y}_2(t)=e^t\big(\boldsymbol{I}+(\boldsymbol{A}-\lambda\boldsymbol{I})t\big)\boldsymbol{v}_2=e^t\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}2 & -4\\ 1 & -2\end{bmatrix}t\right)\boldsymbol{v}_2=\begin{bmatrix}1+2t\\ t\end{bmatrix}e^t.$$

- *Fundamental matrix solution and matrix exponent.* We found that

$$\boldsymbol{\Phi}(t)=\begin{bmatrix}2e^t & (1+2t)e^t\\ e^t & te^t\end{bmatrix}$$

is a fundamental matrix solution for the corresponding linear system. We also have that

$$\boldsymbol{\Phi}(0)=\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix},$$

therefore

$$\boldsymbol{\Phi}^{-1}(0)=\begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix},$$

and hence

$$e^{\boldsymbol{A}t}=\boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(0)=\begin{bmatrix}2e^t & (1+2t)e^t\\ e^t & te^t\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & -2\end{bmatrix}=\begin{bmatrix}(1+2t)e^t & -4te^t\\ te^t & (1-2t)e^t\end{bmatrix}.$$

- *Solving nonhomogeneous system.* We have

$$e^{-\boldsymbol{A}\tau}=\begin{bmatrix}1-2\tau & 4\tau\\ -\tau & 1+2\tau\end{bmatrix}e^{-\tau},$$

therefore,

$$\int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{f}(\tau)\,\mathrm{d}\tau=\begin{bmatrix}t+t^2\\ t+\frac{t^2}{2}\end{bmatrix}.$$

Finally,

$$e^{\boldsymbol{A}t}\begin{bmatrix}t+t^2\\ t+\frac{t^2}{2}\end{bmatrix}=\begin{bmatrix}1-t\\ \frac{1}{2}(2-t)t\end{bmatrix}e^t,$$

whence

$$\boldsymbol{y}(t)=e^{\boldsymbol{A}t}\boldsymbol{y}_0+\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{f}(\tau)\,\mathrm{d}\tau=e^t\begin{bmatrix}1-2t\\ 1-t\end{bmatrix}+\begin{bmatrix}1-t\\ \frac{1}{2}(2-t)t\end{bmatrix}e^t=e^t\begin{bmatrix}1-t-t^2\\ 1-\frac{t^2}{2}\end{bmatrix}.$$

Remark 6. Recall that when we used the variation of the constant method for the second order ODE, there appeared a somewhat arbitrary assumption. To be precise, to solve

$$y'' + py' + qy = f(t)$$

we assumed that

$$y(t) = C_1(t)y_1(t) + C_2(t)y_2(t),$$

where y_1, y_2 are two linearly independent solutions to the homogeneous equation. As one intermediate step we *required* that

$$C_1'y_1 + C_2'y_2 = 0,$$

and that was really difficult to explain why we need it (the usual answer is "because it works.") Now we can see the actual reason for this assumption. First, by introducing $x_1 = y$, $x_2 = x_1' = y'$ I rewrite my equation as a system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -qx_1 - px_2 + f(t) \end{aligned}$$

with the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$$

and the right hand side $\boldsymbol{f}(t) = (0, f(t))^\top$.

Note that a fundamental matrix solution can be obtained as

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix},$$

and therefore the system to determine the unknowns $\boldsymbol{c}(t)$ takes the form

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} C_1'(t) \\ C_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix},$$

where the first equation is *exactly* the same assumption we used before, but now without any mystery.

This simple example shows, among other things, that it is (almost) always preferable to work with systems and not with higher order equations.